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Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane–Emden type

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ABSTRACT

Lane–Emden equation is a nonlinear singular equation in the astrophysics that corresponds to the polytropic models. In this paper, a pseudospectral technique is proposed to solve the Lane–Emden type equations on a semi-infinite domain. The method is based on rational Legendre functions and Gauss–Radau integration. The method reduces solving the nonlinear ordinary differential equation to solve a system of nonlinear algebraic equations. The comparison of the results with the other numerical methods shows the efficiency and accuracy of this method.

1. Introduction

Many science and engineering problems arise in unbounded domains. Different spectral methods have been proposed for solving problems on unbounded domains. The most common method is the use of polynomials that are orthogonal over unbounded domains, such as the Hermite spectral method and the Laguerre method [1–7].

Guo [8–10] proposed a method by mapping the original problem in an unbounded domain to a problem in a bounded domain and then using suitable Jacobi polynomials to approximate the resulting problems.

Another approach is replacing an infinite domain with $[-L, L]$ and a semi-infinite interval with $[0, L]$ by choosing L, sufficiently large. This method is named as domain truncation [11].

Another effective direct approach for solving such problems is based on rational approximations. Christov [12] and Boyd [13,14] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [14] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping them to the Chebyshev polynomials. Guo et al. [15] introduced a new set of rational Legendre functions which are mutually

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orthogonal in $L^2(0,+\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd et al. [16] applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine. Authors of [17] developed two pseudospectral methods based on Fourier series and rational Chebyshev function to solve the Nagumo equation.

Authors of [18–20] applied the spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational Tau method. They obtained the operational matrices of derivative and product of rational Chebyshev and Legendre functions and then applied these matrices together with the Tau method [21–23] to reduce the solution of these problems to the solution of a system of algebraic equations.

In this paper, a pseudospectral technique based on rational Legendre functions is applied to solve nonlinear differential equations, i.e. Lane–Emden and white-dwarf on semi-infinite domain.

Many problems in mathematical physics and astrophysics which occur on semi-infinite interval, are related to the diffusion of heat perpendicular to the parallel planes and can be modeled by the heat equation

$$x^{-k}\frac{d}{dx}\left(x^k\frac{dy}{dx}\right)+f(x)g(y)=h(x),\quad x>0,\ k>0, \tag{1}$$

or equivalently

$$y''+\frac{k}{x}y'+f(x)g(y)=h(x),\quad x>0,\ k>0, \tag{2}$$

where y is the temperature. For the steady-state case and for $k=2, h(x)=0$, this equation is the generalized Emden–Fowler equation [24–26] given by

$$y''+\frac{2}{x}y'+f(x)g(y)=0,\quad x>0, \tag{3}$$

subject to the conditions

$$y(0)=a,\quad y'(0)=b, \tag{4}$$

where $f(x)$ and $g(y)$ are given functions of x and y, respectively.

When $f(x)=1$, Eq. (3) reduces to the Lane–Emden equation which, with specified $g(y)$, was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents.

Several authors have investigated this equation.

Bender et al. [27] proposed a perturbative technique for solving nonlinear differential equations such as Lane–Emden. It consists of replacing nonlinear terms in the Lagrangian such as y^n by $y^{1+\delta}$ and then treating δ as a small parameter.

Shawagfeh [28] applied a nonperturbative approximate analytical solution for the Lane–Emden equation using the Adomian decomposition method. His solution was in the form of a power series. He used Padé approximation method [29] to accelerate the convergence of the power series.

Mandelzweig and Tabakin [30] used the quasilinearization approach to solve Lane–Emden equation. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones, and unlike the perturbation theories is not based on the existence of some kind of small parameters.

Wazwaz [31] employed the Adomian decomposition method [32] with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane–Emden type. Further he used [33] the modified decomposition method for solving analytic treatment of nonlinear differential equations such as Lane–Emden equation. The modified method accelerates the rapid convergence of the series solution, dramatically reduces the size of work and provides the solution by using few iterations only without any need to the so-called Adomian polynomials.

Liao [34] provided an analytic algorithm for Lane–Emden type equations. This algorithm logically contains the well-known Adomian decomposition method. Different from all other analytical techniques, this algorithm itself provides us with a convenient way to adjust convergence regions even without Padé technique.

By the semi-inverse method, He [35] obtained a variational principle for the Lane–Emden equation, which gives much numerical convenience when applying finite element methods or Ritz method.

Parand and Razzaghi [20] presented a numerical technique based on a rational Legendre Tau method to solve higher ordinary differential equations such as Lane–Emden. In their work, the operational matrices of the derivative and product of rational Legendre functions together with the Tau method were utilized to reduce the solution of these physical problems to the solution of systems of algebraic equations.

Ramos [36–38] solved Lane–Emden equation through different methods. He presented linearization methods for singular initial value problems in second order ordinary differential equations such as Lane–Emden. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytically, thus they yield piecewise analytical solutions and globally smooth solutions [36]. Later, he developed piecewise-adaptive decomposition methods for the solution of nonlinear ordinary differential equations. Piecewise-decomposition methods provide series solutions in intervals which are subject to continuity conditions at the end points of each interval and their adaption is based on the use of either a fixed number of approximants and a variable step size, a variable number of approximants and a fixed step size or a variable number of approximants and a variable step size [37]. In [38], series solutions of the Lane–Emden equation based on either a Volterra integral equation formulation or the expansion of the dependent variable in the original ordinary differential equation are presented and compared with series solutions obtained by means of integral or differential equations based on a transformation of the dependent variables.

Yousefi [39] used integral operator and converted Lane–Emden equations to integral equations and then applied Legendre wavelet approximations. In his work properties of Legendre wavelet together with the Gaussian integration method were utilized to reduce the integral equations to the solution of algebraic equations. By his method, the equation was formulated on $[0,1]$.

Chowdhury and Hashim [40] obtained analytical solutions of the generalized Emden–Fowler type equations in the second order ordinary differential equations by homotopy-perturbation method (HPM). This method is a coupling of the perturbation method and the homotopy method. The main feature of the HPM [41] is that it deforms a difficult problem into a set of problems which are easier to solve. HPM yields solutions in convergent series forms with easily computable terms.

Aslanov [42] constructed a recurrence relation for the components of the approximate solution and investigated the convergence conditions for the Emden–Fowler type of equations. He improved the previous results on the convergence radius of the series solution.

Dehghan and Shakeri [43] investigated Lane–Emden equation using the variational iteration method and showed the efficiency and applicability of their procedure for solving this equation. Their technique does not require any discretization, linearization or small perturbations and therefore reduces the volume of computations.

Bataineh et al. [44] obtained analytical solutions of singular initial value problems (IVPs) of the Emden–Fowler type by the homotopy analysis method (HAM). Their solutions contained an auxiliary parameter which provided a convenient way of controlling the convergence region of the series solutions. It was shown that the solutions obtained by the Adomian decomposition method (ADM) and the homotopy-perturbation method (HPM) are only special cases of the HAM solutions.

As one more step in this direction, we use rational Legendre pseudospectral approach to solve Lane–Emden and white-dwarf equations which are nonlinear singular differential equations on semi-infinite interval. The main point of our analysis lies in the fact that there is no reconstruction of the problem on the finite domain. We show that our results have good agreement with exact results, which demonstrate the viability of the new technique. In this sense, this method has the potential to provide a wider applicability. On the other hand, the comparison of the results obtained by this method and the others shows that the new method provides more accurate solutions than those obtained by other methods.

The organization of the paper is as follows:

In Section 2, we explain the formulation of rational Legendre functions required for our subsequent development. In Section 3, after a short introduction to the essentials of Lane–Emden equation, we summarize the application of rational Legendre pseudospectral method for solving Lane–Emden and white-dwarf equations. Then, a comparison is made with the existing methods in the literature. Section 4 is devoted to conclusions.

2. Rational Legendre interpolation

In this section, at first, we introduce rational Legendre functions and express some of their basic properties. More, we approximate a function using Gauss–Radau integration and rational Legendre-Gauss–Radau points.

2.1. Rational Legendre functions

The well-known Legendre polynomials are orthogonal in the interval $[-1,1]$ with respect to the weight function $\rho(y) = 1$ and can be determined with the help of the following recurrence formula:

$$\begin{aligned} &P_0(y) = 1, \quad P_1(y) = y, \\ &P_{n+1}(y) = \left(\frac{2n+1}{n+1}\right) y P_n(y) - \left(\frac{n}{n+1}\right) P_{n-1}(y), \quad n \geqslant 1. \end{aligned} \tag{5}$$

The new basis functions, denoted by $R_n(x)$, are defined as follows:

$$R_n(x) = P_n\left(\frac{x-L}{x+L}\right), \tag{6}$$

where the constant parameter L sets the length scale of the mapping. Boyd [45] offered guidelines for optimizing the map parameter L for rational Chebyshev functions, which is useful for rational Legendre functions, too.

$R_n(x)$ is the nth eigenfunction of the singular Sturm–Liouville problem

$$\frac{(x+L)^2}{L}(xR'_n(x))' + n(n+1)R_n(x) = 0, \quad x \in [0,\infty), \; n = 0,1,2,\ldots \tag{7}$$

and by Eq. (5) satisfies in the following recurrence relation:

$$R_0(x) = 1, \quad R_1(x) = \frac{x - L}{x + L},$$
$$R_{n+1}(x) = \left(\frac{2n+1}{n+1}\right)\left(\frac{x-L}{x+L}\right)R_n(x) - \left(\frac{n}{n+1}\right)R_{n-1}(x), \quad n \geqslant 1. \tag{8}$$

2.2. Function approximation

Let $w(x) = \frac{2L}{(x+L)^2}$ denotes a non-negative, integrable, real-valued function over the interval $\Lambda = [0, \infty)$. We define

$$L^2_w(\Lambda) = \{v : \Lambda \to \mathbb{R} \mid v \text{ is measurable and } \|v\|_w < \infty\}, \tag{9}$$

where

$$\|v\|_w = \left(\int_0^\infty |v(x)|^2 w(x)dx\right)^{\frac{1}{2}}, \tag{10}$$

is the norm induced by the inner product of the space $L^2_w(\Lambda)$,

$$\langle u, v\rangle_w = \int_0^\infty u(x)v(x)w(x)dx. \tag{11}$$

Thus $\{R_n(x)\}_{n\geqslant 0}$ denotes a system which is mutually orthogonal under (11), i.e.,

$$\langle R_n, R_m\rangle_w = \frac{2}{2n+1}\delta_{nm}, \tag{12}$$

where δ_{nm} is the Kronecker delta function. This system is complete in $L^2_w(\Lambda)$. For any function $u \in L^2_w(\Lambda)$ the following expansion holds

$$u(x) = \sum_{k=0}^{+\infty} a_k R_k(x), \tag{13}$$

with

$$a_k = \frac{\langle u, R_k\rangle_w}{\|R_k\|^2_w}. \tag{14}$$

The a_ks are the discrete expansion coefficients associated with the family $\{R_k(x)\}$.

2.3. Rational Legendre interpolation approximation

Canuto et al. [46] and Gottlieb et al. [47] introduced Gauss–Radau integration. Further, Guo et al. [15] introduced rational Legendre-Gauss–Radau points. Let

$$\mathfrak{R}_N = span\{R_0, R_1, \ldots, R_N\}, \tag{15}$$

and $y_j,\ j = 0, 1, \ldots, N$, be the $N+1$ roots of the polynomial $P_{N+1}(x) + P_N(x)$. These points are known as Legendre-Gauss–Radau points. We define

$$x_j = L\frac{1+y_j}{1-y_j}, \quad j = 0, 1, \ldots, N, \tag{16}$$

which is named as rational Legendre-Gauss–Radau nodes. In fact, these points are zeros of the function $R_{N+1}(x) + R_N(x)$. Using Gauss–Radau integration we have:

$$\int_0^\infty u(x)w(x)dx = \int_{-1}^1 u\left(L\frac{1+y}{1-y}\right)\rho(y)dy = \sum_{j=0}^N u(x_j)w_j \quad \forall u \in \mathfrak{R}_{2N}, \tag{17}$$

where

$$w_0 = \frac{2}{(N+1)^2}, \quad w_j = \frac{2L}{(N+1)^2(x_j+L)[R_N(x_j)]^2}, \quad j = 1, \ldots, N, \tag{18}$$

are the corresponding weights with the $N+1$ rational Legendre-Gauss–Radau nodes.

The interpolating function of a smooth function u on a semi-infinite interval is denoted by $P_N u$. It is an element of $\mathfrak{R}_N$ and is defined as

$$P_N u(x) = \sum_{k=0}^N a_k R_k(x). \tag{19}$$

$P_N u$ is the orthogonal projection of u upon $\Re_N$ with respect to the inner product (11) and the norm (10). Thus by the orthogonality of rational Legendre functions we have [15]

$$\langle P_N u - u, R_i \rangle_w = 0 \quad \forall R_i \in \Re_N. \tag{20}$$

To obtain the order of convergence of rational Legendre approximation, at first we define the space

$$H^r_{w,A}(\Lambda) = \{v : v \text{ is measurable and } \|v\|_{r,w,A} < \infty\}, \tag{21}$$

where the norm is induced by

$$\|v\|_{r,w,A} = \left(\sum_{k=0}^{r} \left\| (x+1)^{\frac{r}{2}+k} \frac{d^k}{dx^k} v \right\|_w^2 \right)^{\frac{1}{2}}, \tag{22}$$

and A is the Sturm–Liouville operator as follows:

$$Av(x) = -w^{-1}(x) \frac{d}{dx}\left(x \frac{d}{dx} v(x) \right). \tag{23}$$

We have the following theorem for the convergence:

Theorem 1. *For any $v \in H^r_{w,A}(\Lambda)$ and $r \geqslant 0$,*

$$\|P_N v - v\|_w \leqslant cN^{-r} \|v\|_{r,w,A}. \tag{24}$$

A complete proof of the theorem and discussion on convergence are given in [15].

To apply a pseudospectral approach, we consider the residual $Res(x)$ when the expansion is substituted into the governing equation. It requires that a_k's be selected so that the boundary conditions are satisfied, but make the residual zero at as many (suitable chosen) spatial points as possible.

3. Numerical results

In this section, we apply the pseudospectral approach to find solutions of Lane–Emden and white-dwarf equations. At the first step, by (19), let $P_N y$ be the approximation of y. Thus, our goal is to find the coefficients $a_k, 0 \leqslant k \leqslant N$.

3.1. Lane–Emden equation

Inserting $f(x) = 1$ and $g(y) = y^m$ into (3), we have the standard Lane–Emden equation that corresponds to the polytropic models:

$$y'' + \frac{2}{x} y' + y^m = 0, \quad x > 0. \tag{25}$$

This equation is one of the basic equations in the theory of stellar structure and has been the focus of many studies [27–44]. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. It also describes the variation of density as a function of the radial distance for a polytrope. It was first studied by the astrophysicists Jonathan Homer Lane and Robert Emden, which considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [24,25].

The polytropic theory of stars essentially follows out of thermodynamic considerations, that deal with the issue of energy transport, through the transfer of material between different levels of the star.

The boundary conditions are as follows:

$$y(0) = 1, \quad y'(0) = 0. \tag{26}$$

Physically interesting values of m lie in the interval $[0, 5]$. Exact solutions for Eq. (25) are known only for the values $m = 0, 1$ and 5. For other values of m the Lane–Emden equation is to be integrated numerically. Here, we solved it for $m = 2, 3$ and 4 by pseudospectral method. Let

$$Res(x) = \frac{d^2 P_N y}{dx^2} + \frac{2}{x} \frac{dP_N y}{dx} + P_N^m y, \tag{27}$$

be the residual function of the Lane–Emden equation. $P_N y$ is a good approximation of function y if it is zero on the whole domain. In other words, we should select coefficients a_ks so that the residual function approaches zero on the most of the domain. The pseudospectral scheme for Lane–Emden equation is to find $P_N y \in \Re_N$ such that

$$Res(x_j) = 0, \quad j = 1, \ldots, N-1, \tag{28}$$
$$P_N y(x_0) = 1, \tag{29}$$
$$\left.\frac{dP_N y}{dx}\right|_{x=x_0} = 0, \tag{30}$$

where the x_js are rational Legendre-Gauss–Radau nodes. This generates a set of $N+1$ nonlinear equations that can be solved by Newton method for the unknown coefficients a_ks. In the way that we used the rational Legendre-Gauss–Radau nodes in the equations of finding a_ks, we overcame the singularity behavior at origin.

The first zero of y gives the radius of the star, so y must be computed up to this zero. The gross properties of the star such as, mass, central pressure, binding energy, etc. can be computed through their relations to y. The approximations of the first zero of y obtained by this method and perturbative technique [27] with exact numerical value [48] for $m = 2, 3$ and 4 are listed in Table 1. Compared to the results with exact values, our solution is more accurate.

Table 1
Comparison the first zero of y obtained by present method, perturbation method [27] and exact numerical values [48].

m	N	Present method	Padé approximation	Exact value
2	25	4.35284254	4.3603	4.35287460
	50	4.35286679		
	75	4.35287108		
3	25	6.89678621	7.0521	6.89684862
	50	6.89683601		
	75	6.89684862		
4	25	14.9713392	17.967	14.9715463
	50	14.9714787		
	75	14.9715463		

Table 2
Comparison of $y(x)$, between present method and exact values given by Horedt [48], for $m = 3$.

x	Present method	Exact value
0.000	1.0000000000	1.0000000000
0.100	0.9983350080	0.9983358000
0.500	0.9598341198	0.9598391000
1.000	0.8550456225	0.8550576000
5.000	0.1108089973	0.1108198000
6.000	0.0436802368	0.0437379800
6.800	0.0041551522	0.0041677890
6.896	0.0000358815	0.0000360112

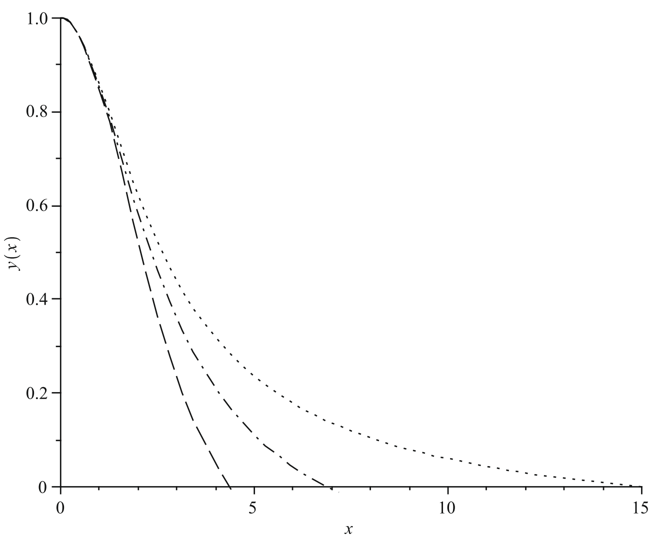

Fig. 1. Lane–Emden graph obtained by present method ($N = 75$) for $m = 2$ (dashed line), $m = 3$ (dashed-dotted line) and $m = 4$ (dotted line).

Table 3
Values a_k of the coefficients of the rational Legendre functions of the Lane–Emden equation for $m = 2(a_0 = 9.9536559038\text{e} - 1)$.

k	a_k	k	a_k	k	a_k	k	a_k	k	a_k
1	−1.3729259001 e−02	16	−6.4945053612 e−02	31	−2.2118596970 e−02	46	5.7473527939 e−03	61	6.3260778427 e−03
2	−2.2363046916 e−02	17	−6.3314059959 e−02	32	−1.9230968288 e−02	47	6.4280815690 e−03	62	5.9122162291 e−03
3	−3.0359563381 e−02	18	−6.1317910929 e−02	33	−1.6467958609 e−02	48	6.9854378897 e−03	63	5.4776814390 e−03
4	−3.7617692117 e−02	19	−5.9007830150 e−02	34	−1.3840554407 e−02	49	7.4267714818 e−03	64	5.0271448752 e−03
5	−4.4078412722 e−02	20	−5.6433317661 e−02	35	−1.1357105639 e−02	50	7.7590336726 e−03	65	4.5657830057 e−03
6	−4.9712512111 e−02	21	−5.3641317111 e−02	36	−9.0243541767 e−03	51	7.9899584843 e−03	66	4.0976307360 e−03
7	−5.4512873241 e−02	22	−5.0676465461 e−02	37	−6.8466881998 e−03	52	8.1267025123 e−03	67	3.6272584033 e−03
8	−5.8489375396 e−02	23	−4.7580355662 e−02	38	−4.8272250346 e−03	53	8.1770924703 e−03	68	3.1580782541 e−03
9	−6.1664806018 e−02	24	−4.4391966214 e−02	39	−2.9670135199 e−03	54	8.1482281386 e−03	69	2.6940768147 e−03
10	−6.4072027898 e−02	25	−4.1146984231 e−02	40	−1.2661648073 e−03	55	8.0477954726 e−03	70	2.2380730777 e−03
11	−6.5751314874 e−02	26	−3.7878383679 e−02	41	2.7700287680 e−04	56	7.8826315689 e−03	71	1.7935049037 e−03
12	−6.6748624253 e−02	27	−3.4615780971 e−02	42	1.6647902495 e−03	57	7.6601021464 e−03	72	1.3626376281 e−03
13	−6.7113673778 e−02	28	−3.1386136871 e−02	43	2.9010338535 e−03	58	7.3866272287 e−03	73	9.4840326562 e−04
14	−6.6898886040 e−02	29	−2.8213122108 e−02	44	3.9898922579 e−03	59	7.0691216053 e−03	74	5.5255837049 e−04
15	−6.6157914630 e−02	30	−2.5117922637 e−02	45	4.9368273289 e−03	60	6.7134797684 e−03	75	1.7757494126 e−04

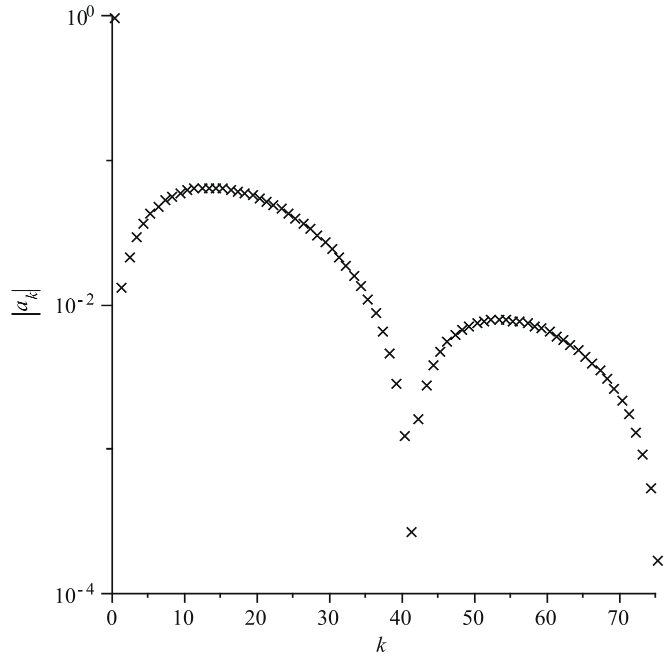

Fig. 2. Absolute values $|a_k|$ of the coefficients of the rational Legendre functions of the Lane–Emden equation for $m = 2$.

Table 4
Values a_k of the coefficients of the rational Legendre functions of the Lane–Emden equation for $m = 3(a_0 = 9.3728978977\text{e} - 1)$.

k	a_k	k	a_k	k	a_k	k	a_k	k	a_k
1	−1.6654612769 e−01	16	1.9238113159 e−02	31	−1.7385385708 e−02	46	−1.9644954922 e−02	61	−3.8760499120 e−03
2	−2.2218571378 e−01	17	1.4573098359 e−02	32	−1.8907569519 e−02	47	−1.8552490801 e−02	62	−3.2492976271 e−03
3	−2.3060889906 e−01	18	1.0417898982 e−02	33	−2.0262517297 e−02	48	−1.7401547612 e−02	63	−2.6940593067 e−03
4	−2.0576045898 e−01	19	6.8691591664 e−03	34	−2.1426054086 e−02	49	−1.6211087370 e−02	64	−2.2075857947 e−03
5	−1.6270952150 e−01	20	3.8889376341 e−03	35	−2.2380801924 e−02	50	−1.4998943435 e−02	65	−1.7863766111 e−03
6	−1.1389032398 e−01	21	1.3683056114 e−03	36	−2.3116072739 e−02	51	−1.3781799112 e−02	66	−1.4260735601 e−03
7	−6.7925437663 e−02	22	−8.2527060898 e−04	37	−2.3627597456 e−02	52	−1.2574823022 e−02	67	−1.1218546644 e−03
8	−2.9747593374 e−02	23	−2.8167455828 e−03	38	−2.3916722092 e−02	53	−1.1391737496 e−02	68	−8.6831514475 e−04
9	−1.3180582498 e−03	24	−4.7058461906 e−03	39	−2.3989762605 e−02	54	−1.0244525085 e−02	69	−6.5986018423 e−04
10	1.7474484703 e−02	25	−6.5601053043 e−03	40	−2.3857034345 e−02	55	−9.1435801572 e−03	70	−4.9056578736 e−04
11	2.7968581113 e−02	26	−8.4149470156 e−03	41	−2.3532205021 e−02	56	−8.0974788687 e−03	71	−3.5456430874 e−04
12	3.2050660043 e−02	27	−1.0278311026 e−02	42	−2.3031396817 e−02	57	−7.1132047501 e−03	72	−2.4588060084 e−04
13	3.1675219015 e−02	28	−1.2137147330 e−02	43	−2.2372674263 e−02	58	−6.1959712857 e−03	73	−1.5880721930 e−04
14	2.8572058962 e−02	29	−1.3964558530 e−02	44	−2.1575283249 e−02	59	−5.3495091342 e−03	74	−8.7714243647 e−05
15	2.4104798894 e−02	30	−1.5726137376 e−02	45	−2.0659291365 e−02	60	−4.5759277553 e−03	75	−2.7413862076 e−05

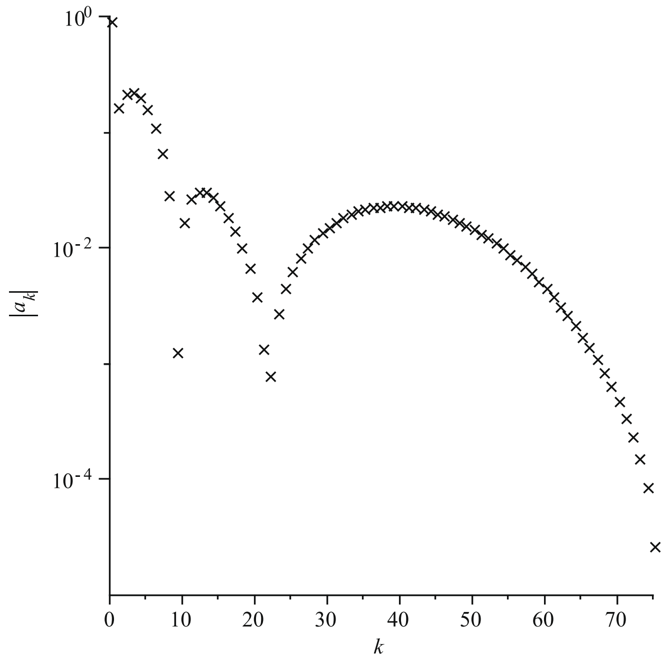

Fig. 3. Absolute values $|a_k|$ of the coefficients of the rational Legendre functions of the Lane–Emden equation for $m = 3$.

Table 5
Values a_k of the coefficients of the rational Legendre functions of the Lane–Emden equation for $m = 4(a_0 = 9.8939043482e - 1)$.

k	a_k	k	a_k	k	a_k	k	a_k	k	a_k
1	−3.0763080674 e−02	16	−3.7596562617 e−02	31	3.8094844466 e−03	46	1.7517841402 e−04	61	−2.2514321634 e−04
2	−4.8223199109 e−02	17	−3.1437532589 e−02	32	3.8185231099 e−03	47	3.8343200566 e−05	62	−2.0164583024 e−04
3	−6.2265759568 e−02	18	−2.5737469630 e−02	33	3.7064636669 e−03	48	−7.2889778446 e−05	63	−1.7866400589 e−04
4	−7.2744235511 e−02	19	−2.0557627114 e−02	34	3.5036088063 e−03	49	−1.6074265159 e−04	64	−1.5660762174 e−04
5	−7.9792808836 e−02	20	−1.5931995757 e−02	35	3.2362329542 e−03	50	−2.2762628605 e−04	65	−1.3578126978 e−04
6	−8.3708382606 e−02	21	−1.1871882952 e−02	36	2.9267125122 e−03	51	−2.7605134157 e−04	66	−1.1636603066 e−04
7	−8.4877997463 e−02	22	−8.3702419233 e−03	37	2.5937085380 e−03	52	−3.0850747661 e−04	67	−9.8472250140 e−05
8	−8.3730221711 e−02	23	−5.4056930803 e−03	38	2.2524539084 e−03	53	−3.2741956094 e−04	68	−8.2118392515 e−05
9	−8.0701686571 e−02	24	−2.9461259198 e−03	39	1.9150426806 e−03	54	−3.3506490197 e−04	69	−6.7282669864 e−05
10	−7.6214127251 e−02	25	−9.5189548136 e−04	40	1.5907853794 e−03	55	−3.3356471489 e−04	70	−5.3878255570 e−05
11	−7.0659120601 e−02	26	6.2144913908 e−04	41	1.2865303895 e−03	56	−3.2482948575 e−04	71	−4.1803381700 e−05
12	−6.4388572856 e−02	27	1.8208439596 e−03	42	1.0070260654 e−03	57	−3.1057612807 e−04	72	−3.0912796493 e−05
13	−5.7709567124 e−02	28	2.6938322980 e−03	43	7.5522515012 e−04	58	−2.9229244395 e−04	73	−2.1066422023 e−05
14	−5.0882460931 e−02	29	3.2870423263 e−03	44	5.3261579606 e−04	59	−2.7127161525 e−04	74	−1.2097301337 e−05
15	−4.4121404005 e−02	30	3.6450441188 e−03	45	3.3948089920 e−04	60	−2.4858819681 e−04	75	−3.8591663085 e−06

Horedt [48] has given exact numerical values of y for some optional x. In Table 2 these values are compared with our results for $m = 3$ and $N = 75$. It shows that our results are highly accurate. The resulting graph of Lane–Emden equation for $N = 75$ and $m = 2, 3$ and 4 is shown in Fig. 1.

Table 3 and Fig. 2 represent the coefficients of the rational Legendre functions obtained by the present method for $N = 75$ and $m = 2$ of the Lane–Emden equation. Table 4, Fig. 3, Table 5 and Fig. 4 give the same information for $m = 3$ and $m = 4$. These tables and figures show that the method has an appropriate convergence rate.

3.2. White-dwarf equation

Inserting $f(x) = 1$ and $g(y) = (y^2 - C)^{\frac{3}{2}}$ into (3) gives the white-dwarf equation

$$y'' + \frac{2}{x}y' + (y^2 - C)^{\frac{3}{2}} = 0, \quad x > 0, \tag{31}$$

which was introduced by [24] in his study of the gravitational potential of the degenerate white dwarf stars. The boundary conditions of this equation are the same as the Lane–Emden boundary conditions in (26) and it has singularity at origin, too. It is interesting to point out that setting $C = 0$ reduces the white-dwarf equation to Lane–Emden equation of index $m = 3$.

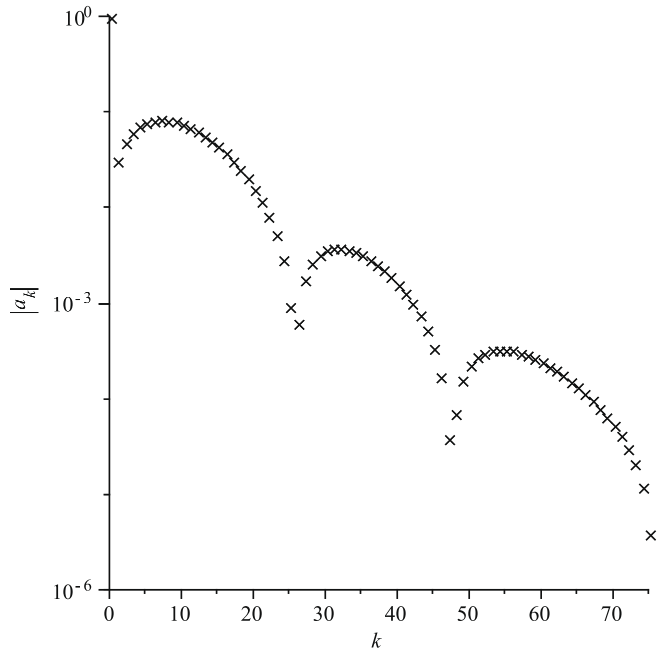

Fig. 4. Absolute values $|a_k|$ of the coefficients of the rational Legendre functions of the Lane–Emden equation for $m = 4$.

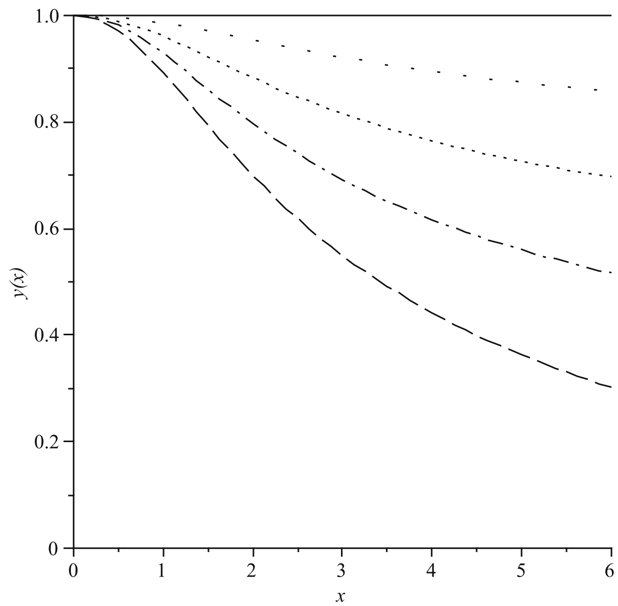

Fig. 5. White-dwarf graph obtained by present method ($N = 7$) for $C = 0.2$ (dashed line), $C = 0.4$ (dashed-dotted line), $C = 0.6$ (dotted line), $C = 0.8$ (spaced-dotted line) and $C = 1.0$ (solid line).

Solving this Eq. (31) is the same as solving the Lane–Emden equation. Let

$$Res(x) = \frac{d^2 P_N y}{dx^2} + \frac{2}{x}\frac{dP_N y}{dx} + (P_N y^2 - C)^{\frac{3}{2}}. \quad (32)$$

So we should find function $P_N y$ that satisfies

$$Res(x_j) = 0, \quad j = 1, \ldots, N-1, \quad (33)$$

$$P_N y(x_0) = 1, \quad (34)$$

$$\left.\frac{dP_N y}{dx}\right|_{x=x_0} = 0. \quad (35)$$

This function can be determined by its coefficients a_ks. These coefficients can be found by solving the set of $N+1$ nonlinear equations in (33)–(35).

Here, we solved this equation for $C = 0.2, 0.4, 0.6, 0.8$ and 1.0. The graph of white-dwarf equation for $N = 7$ and $L = 0.35$ is shown in Fig. 5. The answer for $C = 0.2$ is

$$P_7y(x) = 0.98573 - 0.05409\left(\frac{x-0.35}{x+0.35}\right) - 0.10352\left(\frac{x-0.35}{x+0.35}\right)^2 - 0.17107\left(\frac{x-0.35}{x+0.35}\right)^3 - 0.26409\left(\frac{x-0.35}{x+0.35}\right)^4 - 0.28808\left(\frac{x-0.35}{x+0.35}\right)^5 - 0.17526\left(\frac{x-0.35}{x+0.35}\right)^6 - 0.04390\left(\frac{x-0.35}{x+0.35}\right)^7. \quad (36)$$

4. Conclusion

In the above discussion, we applied the pseudospectral approach to solve nonlinear initial value problems, i.e. Lane–Emden and white-dwarf. Lane–Emden equation occurs in the theory of stellar structure and describes the temperature variation of a spherical gas cloud. The white-dwarf equation appears in the gravitational potential of the degenerate white dwarf stars. The difficulty in this type of equations, due to the existence of singular point at $x = 0$, is overcomed here. In the Lane–Emden equation, the first zero of y is an important point of the function, so we have computed y to this zero. In this paper, this equation is solved for $m = 2, 3$ and 4, which does not have exact solutions. White-dwarf equation is solved for $C = 0.2, 0.4, 0.6, 0.8$ and 1.0. The validity of the method is based on the assumption that it converges by increasing the number of collocation points. Our aim was to apply an accurate and well-conditioned method that give more accurate answers without reformulating the equation to bounded domains. Numerical results indicate the convergence and effectiveness of the present approach.

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References

[1] D. Funaro, Computational aspects of pseudospectral Laguerre approximations, Appl. Numer. Math. 6 (1990) 447–457.
[2] D. Funaro, O. Kavian, Approximation of some diffusion evolution equations in unbounded domains by Hermite function, Math. Comput. 57 (1990) 597–619.
[3] B.Y. Guo, Error estimation of Hermite spectral method for nonlinear partial differential equations, Math. Comput. 68 (1999) 1067–1078.
[4] B.Y. Guo, J. Shen, Laguerre–Galerkin method for nonlinear partial differential equations on a semi-infinite interval, Numer. Math. 86 (2000) 635–654.
[5] Y. Maday, B. Pernaud-Thomas, H. Vandeven, Reappraisal of Laguerre type spectral methods, La Recherche Aerospatiale 6 (1985) 13–35.
[6] J. Shen, Stable and efficient spectral methods in unbounded domains using Laguerre functions, SIAM J. Numer. Anal. 38 (2000) 1113–1133.
[7] H.I. Siyyam, Laguerre Tau methods for solving higher order ordinary differential equations, J. Comput. Anal. Appl. 3 (2001) 173–182.
[8] B.Y. Guo, Gegenbauer approximation and its applications to differential equations on the whole line, J. Math. Anal. Appl. 226 (1998) 180–206.
[9] B.Y. Guo, Jacobi spectral approximation and its applications to differential equations on the half line, J. Comput. Math. 18 (2000) 95–112.
[10] B.Y. Guo, Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations, J. Math. Anal. Appl. 243 (2000) 373–408.
[11] J.P. Boyd, Chebyshev and Fourier Spectral Methods, second ed., Dover, New York, 2000.
[12] C.I. Christov, A complete orthogonal system of functions in $L^2(-\infty,\infty)$ space, SIAM J. Appl. Math. 42 (1982) 1337–1344.
[13] J.P. Boyd, Spectral methods using rational basis functions on an infinite interval, J. Comput. Phys. 69 (1987) 112–142.
[14] J.P. Boyd, Orthogonal rational functions on a semi-infinite interval, J. Comput. Phys. 70 (1987) 63–88.
[15] B.Y. Guo, J. Shen, Z.Q. Wang, A rational approximation and its applications to differential equations on the half line, J. Sci. Comput. 15 (2000) 117–147.
[16] J.P. Boyd, C. Rangan, P.H. Bucksbaum, Pseudospectral methods on a semi-infinite interval with application to the hydrogen atom: a comparison of the mapped Fourier-sine method with Laguerre series and rational Chebyshev expansions, J. Comput. Phys. 188 (2003) 56–74.
[17] M. Dehghan, F. Fakhar-Izadi, Pseudospectral methods for Nagumo equation, Commun. Numer. Meth. Eng, (2009), in press,/doi:10.1002/cnm.1319.
[18] K. Parand, M. Razzaghi, Rational Chebyshev Tau method for solving Volterra's population model, Appl. Math. Comput. 149 (2004) 893–900.
[19] K. Parand, M. Razzaghi, Rational Chebyshev Tau method for solving higher-order ordinary differential equations, Int. J. Comput. Math. 81 (2004) 73–80.
[20] K. Parand, M. Razzaghi, Rational Legendre approximation for solving some physical problems on semi-infinite intervals, Phys. Scr. 69 (2004) 353–357.
[21] M. Dehghan, A. Saadatmandi, A Tau method for the one-dimensional parabolic inverse problem subject to temperature overspecification, Comput. Math. Appl. 52 (2006) 933–940.
[22] A. Saadatmandi, M. Dehghan, Numerical solution of the one-dimensional wave equation with an integral condition, Numer. Meth. Partial Diff. Eqn. 23 (2007) 282–292.
[23] A. Saadatmandi, M. Dehghan, Numerical solution of a mathematical model for capillary formation in tumor angiogenesis via the tau method, Commun. Numer. Meth. Eng. 24 (2008) 1467–1474.
[24] S. Chandrasekhar, Introduction to the Study of Stellar Structure, Dover, New York, 1967.
[25] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962.
[26] O.U. Richardson, The Emission of Electricity from Hot Bodies, Longman's Green and Company, London, 1921.
[27] C.M. Bender, K.A. Milton, S.S. Pinsky, L.M. Simmons Jr., A new perturbative approach to nonlinear problems, J. Math. Phys. 30 (1989) 1447–1455.
[28] N.T. Shawagfeh, Nonperturbative approximate solution for Lane–Emden equation, J. Math. Phys. 34 (1993) 4364–4369.

[29] M. Dehghan, M. Shakourifar, A. Hamidi, The solution of linear and nonlinear systems of Volterra functional equations using Adomian-Pade technique, Chaos Solitons and Fract. 39 (2009) 2509–2521.
[30] V.B. Mandelzweig, F. Tabakin, Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs, Comput. Phys. Commun. 141 (2001) 268–281.
[31] A.M. Wazwaz, A new algorithm for solving differential equations of Lane–Emden type, Appl. Math. Comput. 118 (2001) 287–310.
[32] M. Dehghan, F. Shakeri, The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics, Phys. Scr. 78 (2008) 1–11.
[33] A.M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, Appl. Math. Comput. 173 (2006) 165–176.
[34] S. Liao, A new analytic algorithm of Lane–Emden type equations, Appl. Math. Comput. 142 (2003) 1–16.
[35] J.H. He, Variational approach to the Lane–Emden equation, Appl. Math. Comput. 143 (2003) 539–541.
[36] J.I. Ramos, Linearization techniques for singular initial-value problems of ordinary differential equations, Appl. Math. Comput. 161 (2005) 525–542.
[37] J.I. Ramos, Piecewise-adaptive decomposition methods, Chaos Soliton Fract. (2007), doi:10.1016/j.chaos.2007.09.043.
[38] J.I. Ramos, Series approach to the Lane–Emden equation and comparison with the homotopy perturbation method, Chaos Soliton Fract. 38 (2008) 400–408.
[39] S.A. Yousefi, Legendre wavelets method for solving differential equations of Lane–Emden type, Appl. Math. Comput. 181 (2006) 1417–1422.
[40] M.S.H. Chowdhury, I. Hashim, Solutions of Emden–Fowler equations by homotopy-perturbation method, Nonlinear Anal. Real (2007), doi:10.1016/j.nonrwa.2007.08.017.
[41] M. Dehghan, F. Shakeri, Use of He's homotpy perturbation method for solving a partial differential equation arising in modeling of flow in porous media, J. Porous Media 11 (2008) 765–778.
[42] A. Aslanov, Determination of convergence intervals of the series solutions of Emden–Fowler equations using polytropes and isothermal spheres, Phys. Lett. A 372 (2008) 3555–3561.
[43] M. Dehghan, F. Shakeri, Approximate solution of a differential equation arising in astrophysics using the variational iteration method, New Astron. 13 (2008) 53–59.
[44] A.S. Bataineh, M.S.M. Noorani, I. Hashim, Homotopy analysis method for singular IVPs of Emden–Fowler type, Commun. Nonlinear Sci. Numer. Simul. (2008), doi:10.1016/j.cnsns.2008.02.004.
[45] J.P. Boyd, The optimization of convergence for Chebyshev polynomial methods in an unbounded domain, J. Comput. Phys. 45 (1982) 43–79.
[46] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamic, Springer-Verlag, New York, 1987.
[47] D. Gottlieb, M.Y. Hussaini, S. Orszag, Theory and Applications of Spectral Methods in Spectral Methods for Partial Differential Equations, in: R. Voigt, D. Gottlieb, M.Y. Hussaini (Eds.), SIAM, Philadelphia, 1984.
[48] G.P. Horedt, Polytropes Applications in Astrophysics and Related Fields, Kluwer Academic Publishers, Dordrecht, 2004.